Fuzzy Sets

by
L.A. Zadeh

November 16, 1964

Report No. 64-44
AF-AFOSR-139-64
NSF GP-2413

ELECTRONICS RESEARCH LABORATORY

UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

Requests for additional copies by Agencies of the Department of Defense, their contractors, and other Government agencies should be directed to the:

DEFENSE DOCUMENTATION CENTER (DDC)
CAMERON STATION
ALEXANDRIA, VIRGINIA 22314

Department of Defense contractors must be established for DDC services or have their "need-to-know" certified by the cognizant military agency of their project or contract.

Electronics Research Laboratory
University of California
Berkeley, California
Report No. 64-44

FUZZY SETS

by

L. A. Zadeh

This work was supported in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy and U.S. Air Force) under Grant No. AF-AFOSR-139-64 and by the National Science Foundation under Grant GP-2413.

November 16, 1964

ABSTRACT

A fuzzy set is a class of objects without a precisely defined criterion of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.

I. INTRODUCTION

More often than not, the classes of objects we deal with in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc., as its members, and clearly excludes such objects as rocks, fluids, plants, etc. However, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a number such as 10 in relation to the "class" of all real numbers which are much greater than 1.

Clearly, the "class of all real numbers which are much greater than 1," or "the class of beautiful women," or "the class of tall men," do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, abstraction and communication of information.

The purpose of this note is to explore in a preliminary way some of the basic properties and implications of a concept which may be of use in dealing with "classes" of the type cited above. The concept in question is that of a <u>fuzzy set</u>,* that is, a "set" without a dichotomous criterion of membership. As will be seen in the sequel, the notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing.

We begin the discussion of fuzzy sets with several basic definitions

---

*An application of this concept to the formulation of a class of problems in pattern classification is described in RAND Memorandum RM-4307-PR, "Abstraction and Pattern Classification," by R. Bellman, R. Kalaba and L. A. Zadeh, October, 1964.

II. DEFINITIONS

Let X be a space of points (objects), with a generic element of X denoted by x. Thus, $X = \{x\}$.

A fuzzy set (class) A in X is characterized by a membership (characteristic) function $f_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$,* with the value of $f_A(x)$ at x representing the "grade of membership" of x in A. Thus, the nearer the value of $f_A(x)$ to unity, the higher the grade of membership of x in A. When A is a set in the ordinary sense of the term, its membership function can take on only two values 0 and 1, with $f_A(x) = 1$ or 0 according as x does or does not belong to A. Thus, in this case $f_A(x)$ reduces to the familiar characteristic function of a set A. (When there is a need to differentiate between such sets and fuzzy sets, the sets with two-valued characteristic functions will be referred to as ordinary sets or simply sets.)

Example. Let X be the real line R^1 and let A be a fuzzy set of numbers which are much greater than 1. Then, one can give a precise, albeit subjective, characterization of A by specifying $f_A(x)$ as a function on R^1. Representative values of such a function might be: $f_A(0) = 0$; $f_A(1) = 0$; $f_A(5) = 0.01$; $f_A(10) = 0.2$; $f_A(100) = 0.95$; $f_A(500) = 1$.

The following definitions involving fuzzy sets are obvious extensions of the corresponding definitions for ordinary sets.

A fuzzy set is empty if and only if its membership function is identically zero on X.

---

*In a more general setting, the range of the membership function can be taken to be a suitable partially ordered set P. For our purposes, it is convenient and sufficient to restrict the range of f to the unit interval. If the values of $f_A(x)$ are interpreted as truth values, the latter case corresponds to a logic with a (possibly) continuum of truth values in the interval $[0, 1]$.

Two fuzzy sets A and B are <u>equal</u>, written as $A = B$, if and only if $f_A(x) = f_B(x)$ for all x in X. [In the sequel, instead of writing $f_A(x) = f_B(x)$ for all x in X, we shall write more simply $f_A = f_B$.]

The <u>complement</u> of a fuzzy set A is denoted by A' and is defined by

$$f_{A'} = 1 - f_A . \tag{1}$$

As in the case of ordinary sets, the notion of containment or belonging plays a central role in the case of fuzzy sets. This notion and the related notions of union and intersection are defined as follows.

<u>Containment</u>. A is <u>contained in</u> B (or, equivalently, A is a <u>subset of</u> B, or A is <u>smaller than or equal to</u> B) if and only if $f_A \le f_B$. In symbols

$$A \subset B \Longleftrightarrow f_A \le f_B . \tag{2}$$

<u>Union</u>. The <u>union</u> of two fuzzy sets A and B with respective membership functions $f_A(x)$ and $f_B(x)$ is a fuzzy set C, written as $C = A \cup B$, whose membership function is related to those of A and B by

$$f_C(x) = \mathrm{Max}[\, f_A(x),\ f_B(x)], \qquad x \in X \tag{3}$$

or, in abbreviated form

$$f_C = f_A \vee f_B . \tag{4}$$

<u>Comment.</u> A more intuitively appealing way of defining the union is the following: The union of A and B is the smallest fuzzy set containing both A and B. More precisely, if D is any fuzzy set which contains both A and B, then it also contains the union of A and B.

To show that this definition is equivalent to (3), we note, first, that C as defined by (3) contains both A and B, since

$$\mathrm{Max}[\, f_A,\ f_B] \ge f_A$$

and

$$\mathrm{Max}[\, f_A,\ f_B] \ge f_B .$$

Furthermore, if D is any fuzzy set containing both A and B, then

$$f_D \geq f_A$$

$$f_D \geq f_B$$

and hence

$$f_D \geq \mathrm{Max}[f_A, f_B] = f_C$$

which implies that $C \subset D$. Q. E. D.

The notion of an intersection of fuzzy sets can be defined in an analogous manner. Specifically:

<u>Intersection.</u> The <u>intersection</u> of two fuzzy sets A and B with respective membership functions $f_A(x)$ and $f_B(x)$ is a fuzzy set C, written as $C = A \cap B$, whose membership function is related to those of A and B by

$$f_C(x) = \mathrm{Min}[f_A(x), f_B(x)], \qquad x \in X, \tag{5}$$

or, in abbreviated form

$$f_C = f_A \wedge f_B. \tag{6}$$

As in the case of the union, it is easy to show that the intersection of A and B is the <u>largest</u> fuzzy set which is contained in both A and B. As in the case of ordinary sets, A and B are <u>disjoint</u> if $A \cap B$ is empty.

The intersection and union of two fuzzy sets in R^1 are illustrated in Fig. 1. The membership function of the union is comprised of curve segments 1 and 2; that of the intersection is comprised of segments 3 and 4 (heavy lines).

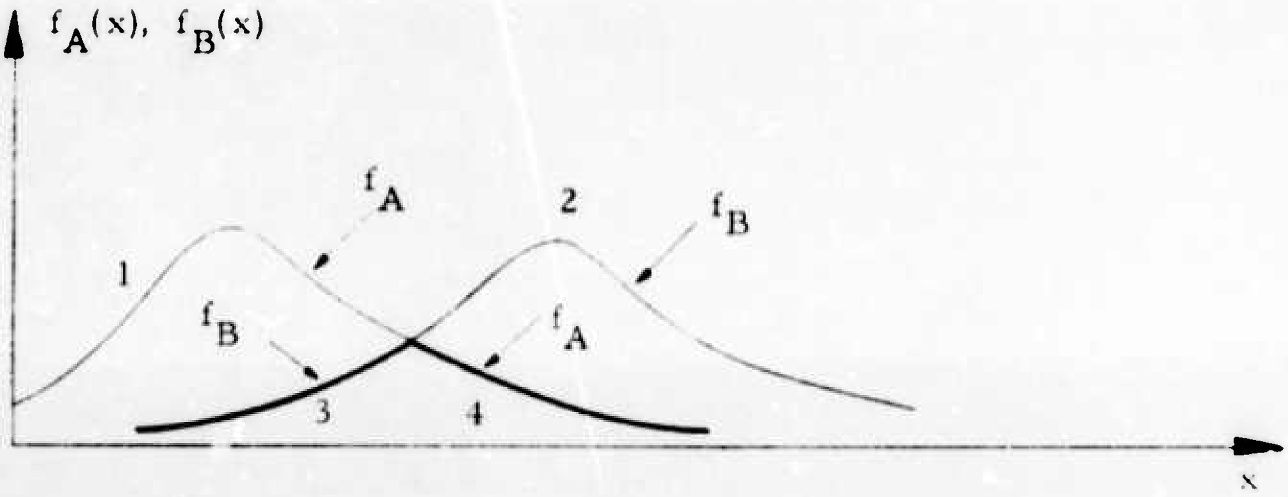

Fig. 1.

III. SOME PROPERTIES OF ∪, ∩ AND COMPLEMENTATION

With the operations of union, intersection and complementation defined as in (3), (5), and (1), it is easy to extend many of the basic identities which hold for ordinary sets to fuzzy sets. As examples, we have

$$\left.\begin{aligned}(A \cup B)' &= A' \cap B' \\ (A \cap B)' &= A' \cup B'\end{aligned}\right\} \text{ De Morgan's laws} \tag{7}$$

$$\left.\begin{aligned}C \cap (A \cup B) &= (C \cap A) \cup (C \cap B) \quad (9)\\ C \cup (A \cap B) &= (C \cup A) \cap (C \cup B) \quad (10)\end{aligned}\right\} \text{ Distributive laws.}$$

These and similar equalities can readily be established by showing that the corresponding relations for the membership functions of A B and C are identities. For example, in the case of (7), we have

$$1 - \mathrm{Max}[f_A, f_B] = \mathrm{Min}[1 - f_A, 1 - f_B] \tag{11}$$

which can be easily verified to be an identity by testing it for the two possible cases: $f_A(x) > f_B(x)$ and $f_A(x) < f_B(x)$.

Similarly, in the case of (10), the corresponding relation in terms of f_A, f_B and f_C is:

$$\mathrm{Max}\left[f_C, \mathrm{Min}[f_A, f_B]\right] = \mathrm{Min}\left[\mathrm{Max}[f_C, f_A], \mathrm{Max}[f_C, f_B]\right] \tag{12}$$

which can be verified to be an identity by considering the six cases:

$f_A(x) > f_B(x) > f_C(x)$, $f_A(x) > f_C(x) > f_B(x)$, $f_B(x) > f_A(x) > f_C(x)$,

$f_B(x) > f_C(x) > f_A(x)$, $f_C(x) > f_A(x) > f_B(x)$, $f_C(x) > f_B(x) > f_A(x)$.

<u>An Interpretation for Unions and Intersections</u>

In the case of ordinary sets, a set C which is expressed in terms of a family of sets $A_1, \ldots, A_i, \ldots, A_n$ through the connectives $\cup$ and $\cap$, can be represented as a network of switches $\alpha_1, \ldots, \alpha_n$, with $A_i \cap A_j$ and $A_i \cup A_j$ corresponding, respectively, to series and parallel combinations of α_i and α_j. In the case of fuzzy sets, one can give an analogous interpretation in terms of sieves. Specifically, let $f_i(x)$, $i = 1, \ldots, n$, denote the value of the membership function of A at x. Associate with $f_i(x)$ a sieve $S_i(x)$ whose meshes are of size $f_i(x)$. Then, $f_i(x) \vee f_j(x)$ and $f_i(x) \wedge f_j(x)$, correspond, respectively, to parallel and series combinations of $S_i(x)$ and $S_j(x)$, as shown in Fig. 2.

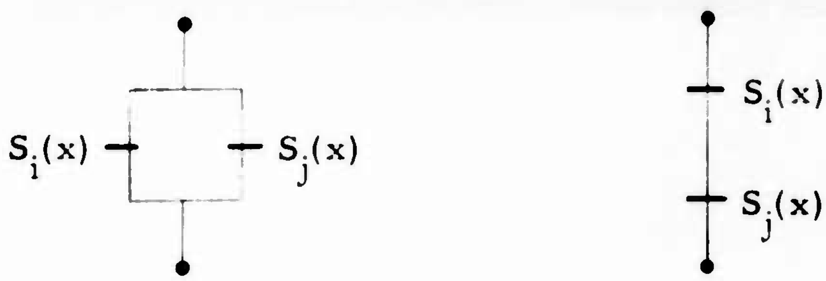

Fig. 2.

More generally, a well-formed expression involving $A_1, \ldots, A_n$, $\cup$ and $\cap$ corresponds to a network of sieves $S_1(x), \ldots, S_n(x)$ which can be found by the conventional synthesis techniques for switching circuits. As a very simple example,

$$C = [(A_1 \cup A_2) \cap A_3] \cup A_4 \tag{13}$$

corresponds to the network shown in Fig. 3.

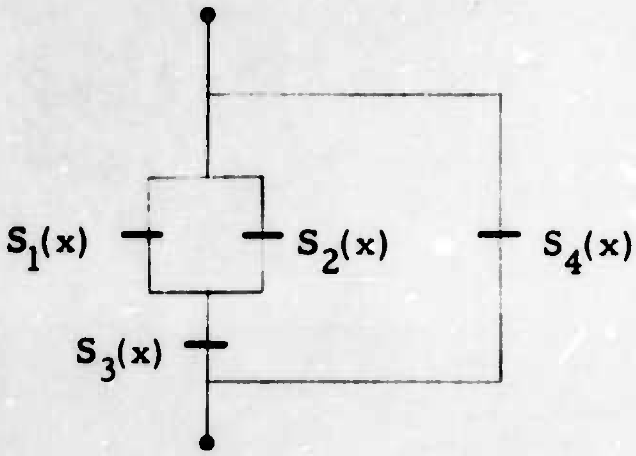

Fig. 3.

Note that the mesh sizes of the sieves in the network depend on x and that the network as a whole is equivalent to a single sieve whose meshes are of size $f_C(x)$.

IV. ALGEBRAIC OPERATIONS ON FUZZY SETS

In addition to the operations of union and intersection, one can define a number of other ways of forming combinations of fuzzy sets and relating them to one another. Among the more important of these are the following.

Albegraic product. The algebraic product of A and B is denoted by AB and is defined in terms of the membership functions of A and B by the relation

$$f_{AB} = f_A f_B . \tag{14}$$

Clearly,

$$AB \subset A \cap B. \tag{15}$$

Algebraic sum. The algebraic sum of A and B is denoted by $A + B$ and is defined by

$$f_{A+B} = f_A + f_B \tag{16}$$

provided the sum $f_A + f_B$ is less than or equal to unity. Thus, unlike the algebraic product, the algebraic sum is meaningful only when the condition $f_A(x) + f_B(x) \leq 1$ is satisfied for all x.

Convex combination. By a convex combination of two vectors f and g is usually meant a linear combination of f and g of the form $\lambda f + (1 - \lambda)g$, in which $0 \leq \lambda \leq 1$. This mode of combining f and g can be generalized to fuzzy sets in the following manner.

Let A, B and Λ be arbitrary fuzzy sets. The convex combination of A, B and Λ is denoted by $(A, B; \Lambda)$ and is defined by the relation

$$(A, B; \Lambda) = \Lambda A + \Lambda' B \tag{17}$$

where Λ' is the complement of Λ. Written out in terms of membership functions, (17) reads

$$f_{(A,B;\Lambda)}(x) = f_{\Lambda}(x)\, f_A(x) + (1 - f_{\Lambda}(x)\, f_B(x), \quad x \in X. \tag{18}$$

A basic property of the convex combination of A, B and Λ is expressed by

$$A \cap B \subset (A, B; \Lambda) \subset A \cup B \qquad \text{for all } \Lambda. \tag{19}$$

This property is an immediate consequence of the inequalities

$$\mathrm{Min}[f_A(x), f_B(x)] \leq \lambda f_A(x) + (1 - \lambda) f_B(x) \leq \mathrm{Max}[f_A(x), f_B(x)], \quad x \in X \tag{20}$$

which hold for all λ in $[0, 1]$. It is of interest to observe that, given any fuzzy set C satisfying $A \cap B \subset C \subset A \cup B$, one can always find a fuzzy set Λ such that $C = (A, B; \Lambda)$. The membership function of this set is given by

$$f_{\Lambda}(x) = \frac{f_C(x) - f_B(x)}{f_A(x) - f_B(x)}, \quad x \in X. \tag{21}$$

Fuzzy Sets Induced by Mappings

Let T be a mapping from X to a space Y. Let B be a fuzzy set in Y with characteristic function $f_B(y)$. The inverse mapping T^{-1} induces a fuzzy set A in X whose membership function is defined by

$$f_A(x) = f_B(y), \qquad y \in Y \tag{22}$$

for all x in X which are mapped by T into y.

Consider now a converse problem in which A is a given fuzzy set in X, and T, as before, is a mapping from X to Y. The question is: What is the membership function for the fuzzy set B in Y which is induced by this mapping?

If T is not one-one, then an ambiguity arises when two or more distinct points in X, say x_1 and x_2, with different grades of membership in A, are mapped into the same point y in Y. In this case, the question is: What grade of membership in B should be assigned to y?

To resolve this ambiguity, we agree to assign the larger of the two grades of membership to y. More generally, the membership function for B will be defined by

$$f_B(y) = \operatorname*{Max}_{x \in T^{-1}(y)} f_A(x), \qquad y \in Y \tag{23}$$

where $T^{-1}(y)$ is the set of points in X which are mapped into y by T.

V. CONVEXITY

As will be seen in the sequel, the notion of convexity can readily be extended to fuzzy sets in such a way as to preserve many of the properties which it has in the context of ordinary sets. This notion appears to be particularly useful in applications involving pattern classification, optimization and related problems.

In what follows, we assume for concreteness that X is a real Euclidean space E^n.

Definitions

Convexity. A fuzzy set A is convex if and only if the sets Γ_α defined by

$$\Gamma_\alpha = \{ x \mid f_A(x) \geq \alpha \} \tag{24}$$

are convex for all α in the interval $(0, 1]$.

An alternative and more direct definition of convexity is the following:* A is convex if and only if

$$f_A[\lambda x_1 + (1 - \lambda) x_2] \geq \mathrm{Min}[f_A(x_1), f_A(x_2)] \tag{25}$$

for all x_1 and x_2 in X and all λ in $[0, 1]$. Note that this definition does not imply that $f_A(x)$ must be a convex function of x. This is illustrated in Fig. 4 for $n = 1$.

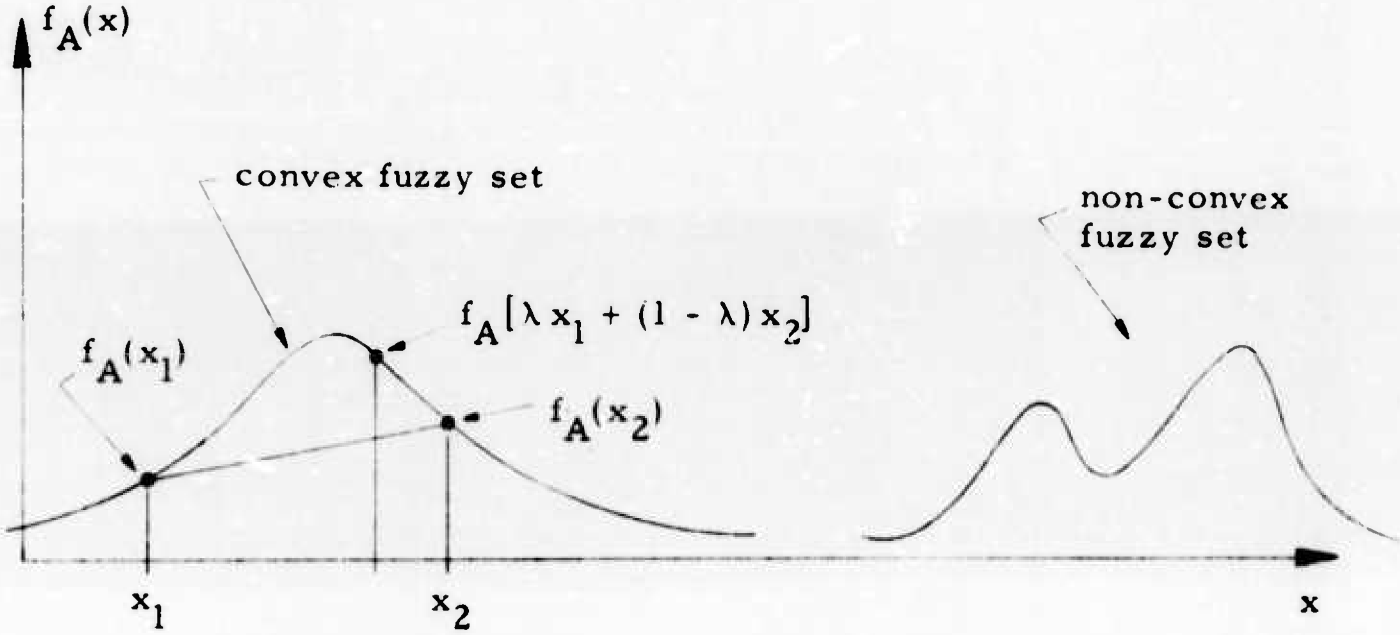

Fig. 4.

* This way of expressing convexity was suggested to the writer by his colleague, E. Berlekamp.

To show the equivalence between the above definitions note that if A is convex in the sense of the first definition and $\alpha = f_A(x_1) \leq f_A(x_2)$, then $x_2 \epsilon \Gamma_\alpha$ and $\lambda x_1 + (1 - \lambda) x_2 \epsilon \Gamma_\alpha$ by the convexity of Γ_α. Hence $f_A[\lambda x_1 + (1 - \lambda) x_2] \geq \alpha = f_A(x_1) = \mathrm{Min}[f_A(x_1), f_A(x_2)]$.

Conversely, if A is convex in the sense of the second definition and $\alpha = f_A(x_1)$, then Γ_α may be regarded as the set of all points x_2 for which $f_A(x_2) \geq f_A(x_1)$. In virtue of (25), every point of the form $\lambda x_1 + (1 - \lambda) x_2$, $0 \leq \lambda \leq 1$, is also in Γ_α and hence Γ_α is a convex set. Q. E. D.

A basic property of convex fuzzy sets is expressed by the

Theorem. If A and B are convex, so is their intersection.

Proof. Let $C = A \cap B$. Then

$$f_C[\lambda x_1 + (1 - \lambda) x_2] = \mathrm{Min}\Big[f_A[\lambda x_1 + (1 - \lambda) x_2],\ f_B[\lambda x_1 + (1 - \lambda) x_2]\Big]. \tag{26}$$

Now, since A and B are convex

$$f_A[\lambda x_1 + (1 - \lambda) x_2] \geq \mathrm{Min}[f_A(x_1), f_A(x_2)] \tag{27}$$

$$f_B[\lambda x_1 + (1 - \lambda) x_2] \geq \mathrm{Min}[f_B(x_1), f_B(x_2)]$$

and hence

$$f_C[\lambda x_1 + (1 - \lambda) x_2] \geq \mathrm{Min}\Big[\mathrm{Min}[f_A(x_1), f_A(x_2)],\ \mathrm{Min}[f_B(x_1), f_B(x_2)]\Big] \tag{28}$$

or equivalently

$$f_C[\lambda x_1 + (1 - \lambda) x_2] \geq \mathrm{Min}\Big[\mathrm{Min}[f_A(x_1), f_B(x_1)],\ \mathrm{Min}[f_A(x_2), f_B(x_2)]\Big] \tag{29}$$

and thus

$$f_C[\lambda x_1 + (1 - \lambda) x_2] \geq \mathrm{Min}[f_C(x_1), f_C(x_2)]. \quad \text{Q. E. D.} \tag{30}$$

Boundedness and strong convexity. A fuzzy set A is bounded if and only if the sets $\Gamma_\alpha = \{ x \mid f_A(x) \geq \alpha \}$ are bounded for all $\alpha > 0$; that is, for every $\alpha > 0$ there exists an $R(\alpha)$ such that $\| x \| \leq R(\alpha)$ for all x in Γ_α.

A fuzzy set A is strictly convex if the sets Γ_α, $0 < \alpha \leq 1$ are strictly convex (that is, if the midpoint of any two distinct points in Γ_α lies in the interior of Γ_α). Note that this definition reduces to that of strict convexity for ordinary sets when A is such a set.

A fuzzy set A is strongly convex if, for any two distinct points x_1 and x_2, and any λ in the open interval (0, 1)

$$f_A[\lambda x_1 + (1 - \lambda) x_2] > \text{Min}[f_A(x_1), f_A(x_2)] .$$

Note that strong convexity does not imply strict convexity or vice-versa. Note also that if A and B are bounded, so is their union and intersection. Similarly, if A and B are strictly (strongly) convex, their intersection is strictly (strongly) convex.

Let A be a convex fuzzy set and let $M = \sup_x f_A(x)$. If A is bounded, then either M is attained for some x, say x_0, or there is at least one point x_0 at which M is essentially attained in the sense that, for each $\varepsilon > 0$, every spherical neighborhood of x_0 contains points in the set $Q(\varepsilon) = \{ x \mid M - f_A(x) \leq \varepsilon \}$. In particular, if A is strongly convex and x_0 is attained, then x_0 is unique. In the sequel, M will be referred to as the maximal grade in A.

Separation of Convex Fuzzy Sets

The classical separation theorem for ordinary convex sets states, in essence, that if A and B are disjoint convex sets, then there exists a separating hyperplane H such that A is on one side of H and B is on the other side.

It is natural to inquire if this theorem can be extended to convex fuzzy sets, without requiring that A and B be disjoint, since the condition of disjointness is much too restrictive in the case of fuzzy sets. It turns out, as will be seen in the sequel, that the answer to this question is in the affirmative.

As a preliminary, we shall have to make a few definitions. Specifically, let A and B be two bounded fuzzy sets and let H be a hypersurface in E^n defined by an equation $h(x) = 0$, with all points for which $h(x) > 0$ being on one side of H and all points for which $h(x) < 0$ being on the other side. Let K_H be a number dependent on H such that $f_A(x) \leq K_H$ on one side of H and $f_B(x) \leq K_H$ on the other side. Let M_H be Inf K_H. The number $D_H = 1 - M_H$ will be called the *degree of separation of A and B by H*.

In general, one is concerned not with a given hypersurface H, but with a family of hypersurfaces $\{H_\lambda\}$, with λ ranging over, say, E^m. The problem, then, is to find a member of this family which realizes the highest possible degree of separation.

A special case of this problem is one where the H_λ are hyperplanes in E^n, with λ ranging over E^n. In this case, we define the *degree of separability* of A and B by the relation

$$D = 1 - \bar{M} \qquad (31)$$

where

$$\bar{M} = \operatorname*{Inf}_{H} M_H \qquad (32)$$

with the subscript λ omitted for simplicity.

Among the various assertions that can be made concerning D, the following statement* is, in effect, an extension of the separation theorem to convex fuzzy sets.

*This statement is a modified version of a separation theorem suggested by E. Berlekamp. (The statement is trivially true if M is equal to M_A or M_B.)

Theorem. Let A and B be bounded convex fuzzy sets in E^n, with maximal grades M_A and M_B, respectively [$M_A = \sup_x f_A(x)$, $M_B = \sup_x f_B(x)$] . Let M be the maximal grade for the intersection $A \cap B$ ($M = \sup_x \mathrm{Min}[f_A(x), f_B(x)]$). If $M < M_A$, $M < M_B$ then $D = 1 - M$.

Comment. In plain words, the theorem states that the highest degree of separation of two convex fuzzy sets A and B that can be achieved with a hyperplane in E^n is one minus the maximal grade in the intersection $A \cap B$, provided it is smaller than the maximal grades in A and B. This is illustrated in Fig. 5 for $n = 1$.

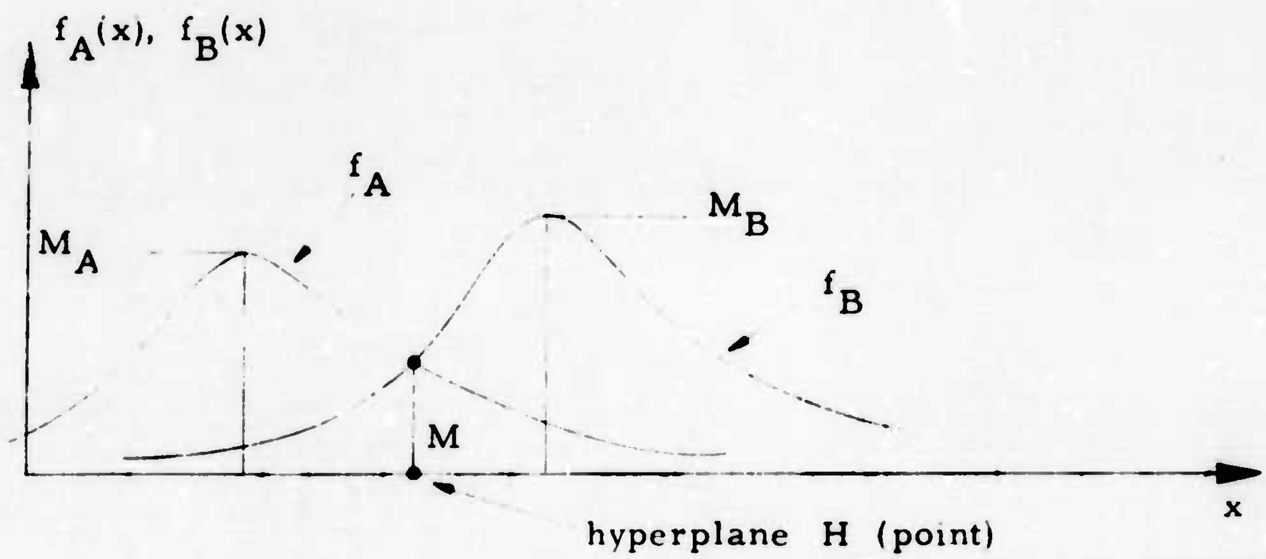

Fig. 5.

Proof. Consider the convex sets $\Gamma_A = \{x \mid f_A(x) > M\}$ and $\Gamma_B = \{x \mid f_B(x) > M\}$. These sets are non-empty and disjoint, for if they were not there would be a point, say u, such that $f_A(u) > M$ and $f_B(u) > M$, and hence $f_{A \cap B}(u) > M$, which contradicts the assumption that $M = \sup_x f_{A \cap B}(x)$.

Since Γ_A and Γ_B are disjoint, by the separation theorem for ordinary convex sets there exists a hyperplane H such that Γ_A is on one side of H (say, the plus side) and Γ_B is on the other side (the minus side). Furthermore, by the definitions of Γ_A and Γ_B, for all points on the minus side of H, $f_A(x) \leq M$, and for all points on the plus side of H, $f_B(x) \leq M$.

It remains to be shown that there does not exist an M', $M' < M$, such that for some H, $f_A(x) \leq M'$ on one side of H and $f_B(x) \leq M'$ on the other side.

Suppose such an M' and H did exist. Then the set $\Gamma = \{x \mid f_{A \cap B}(x) > M'\}$ would be non-empty, for otherwise $M = \sup_x f_{A \cap B}(x)$ would be a contradiction. But if for some x, say w,

$$f_{A \cap B}(w) > M'$$

then

$$f_A(w) > M$$

and

$$f_B(w) > M'$$

since $f_{A \cap B} = \mathrm{Min}[f_A(w), f_B(w)]$. Clearly, the existence of the point w is inconsistent with one or the other of the inequalities: $f_A(x) \leq M'$ on one side of H, and $f_B(x) \leq M'$ on the other side of H. This shows, by contradiction, that there does not exist a hyperplane H for which the degree of separation of A and B is larger than 1 - M. Q. E. D.

The separation theorem for convex fuzzy sets appears to be of particular relevance to the problem of pattern discrimination. Its application to this class of problems as well as to problems of optimization will be explored in subsequent notes on fuzzy sets and their properties.